Stochastic heat equations on moving domains

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Abstract

In this paper, we establish the well-posedness of stochastic heat equations on moving domains, which amounts to a study of infinite dimensional interacting systems. The main difficulty is to deal with the problems caused by the time-varying state spaces and the interaction of the particle systems. The interaction still occurs even in the case of additive noise. This is in contrast to stochastic heat equations in a fixed domain.

Introduction

We are concerned with the well-posedness of stochastic heat equations driven by multiplicative noise on a moving domain, which is given as follows,

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \Delta_t u(t,x) + \sigma\big(t, u(t,x)\big)\dfrac{\partial^2 W(t,x)}{\partial t \partial x} & \text{for } (t,x) \in \mathcal{O}_T, \\ u(t,x) = 0 \quad \text{for } x \in \partial I_t, \\ u(0,x) = u_0(x) \quad \text{for } x \in I_0. \end{cases} \tag{1}$$

Here $\mathcal{O}_T = \underset{0 \le t \le T}{\cup} \{t\} \times (0, a_t)$ is a non-cylindrical time-space domain, $a : [0, T] \to (0, +\infty)$ is a continuously differentiable function, Δ_t stands for the Dirichlet Laplacian operator on $I_t := (0, a_t)$ with boundary $\partial I_t = \{0, a_t\}$, and $\frac{\partial^2 W(t,x)}{\partial t \partial x}$ stands for a space-time noise specified later.

Introduction

The classical heat equation, i.e. equation (1) with $\sigma \equiv 0$ and $I_t \equiv I_0$, describes the evolution of the heat flow on a fixed domain. The solution $u(t, x)$ is the heat density at position $x \in I_0$ and time $t \in [0, T]$ with initial density $u_0(x)$. In the stochastic setting, the noise term represents a random internal/external heat source. The equation (1) describes the time evolution of the heat density in a domain moving with time and with a time-dependent random heat source.

Introduction

To solve deterministic PDEs on time-varying domains, there are mainly two approaches in the literature. One is called the "diffeomorphism method", which transforms the original PDEs on the moving domain into PDEs on a flat domain via a family of diffeomorphisms. Although the new PDEs are on a fixed domain, they have much higher nonlinear (complex) terms than the original PDEs. The other is called the "penalty method". The PDEs are solved by adding penalty terms and taking the limits.
For the stochastically-forced equations on time-varying domains, the problem becomes much more delicate because of the singularities introduced by the noise term. There are very few results on the well-posedness of stochastic partial differential equations on moving domains.

We refer the reader to a recent paper [3] by some of the authors of this paper, in which the authors established the well-posedness of stochastic 2D Navier-Stokes equations on a time-dependent domain driven by an additive noise. In that paper, the "diffeomorphism method" has been adopted. However, using this method, it seems very difficult to build the Itô formula for the solutions, and to solve the case of multiplicative noises. Similar problems arise when one applies the "penalty method". The methods mentioned above are suitable to solve PDEs on time-varying domains, but it seems less ineffective when one uses them to deal with the case of the stochastically-forced equations.

Introduction

The purpose of this paper is to establish the well-posedness of stochastic heat equations driven by multiplicative noise on one dimensional moving domains. We use the Galerkin method, commonly used for (stochastic) PDEs on fixed domains, but it seems that this is the first time to use the method to handle stochastic equations on moving domains, even for deterministic equations on moving domains. Compared with the case of fixed domains, new essential difficulties appear. For example, the eigenbasis of time-dependent Laplacian Δ_t is dependent on t.

Introduction

Using the evolving eigenbasis of time-dependent Laplacian, it turns out that the problem becomes a study of the well-posedness of an infinite interacting particle system. We then use finite dimensional approximations. Through proving the tightness of the laws of the approximating solutions, we first obtain the existence of a probabilistic weak solution to equation (1) and then the well-posedness of strong solutions by combining the pathwise uniqueness and Yamada-Watanable theorem. The main difficulty is to deal with the problem caused by the time-varying state space and the interaction of the particle systems. We notice that the corresponding system is still interacting even in the case of additive noise. This is in contrast to stochastic heat equations on a fixed domain.

We assume that the domain I_t deforms in time t in a continuously differentiable way, more precisely,

Assumption I.
There exists a $\mathcal{C}^1$ function $a : [0, T] \to (0, \infty)$ such that $I_t = (0, a_t)$, $\forall\ t \in [0, T]$.
In the following, the derivative of a_t is denoted by a'_t.

Let $\mathbb{L}^2(I_t)$ be the space of all square-integrable functions on I_t and $\mathbb{H}_0^1(I_t)$ the closure of the space of all smooth functions compactly supported in I_t under the norm

$$\|u\|_{\mathbb{H}_0^1(I_t)} = \|\partial_x u\|_{\mathbb{L}^2(I_t)}.$$

For simplicity, $\forall\ t \geq 0$, we write $\|\cdot\|_{\mathbb{L}^2(I_t)}$ as $|\cdot|_t$, $(\cdot,\cdot)_{\mathbb{L}^2(I_t)}$ as $(\cdot,\cdot)_t$, $\|\cdot\|_{\mathbb{H}_0^1(I_t)}$ as $\|\cdot\|_t$, and $(\cdot,\cdot)_{\mathbb{H}_0^1(I_t)}$ as $\langle\cdot,\cdot\rangle_t$. Sometimes we will also write $\mathbb{L}^2(I_t)$ and $\mathbb{H}_0^1(I_t)$ as $\mathbb{H}_t$ and $\mathbb{V}_t$ respectively.

Now we introduce the function spaces that the solution belongs to. Fix a constant L such that $L > \sup_{0 \leq t \leq T} a_t$, the function space $\mathbb{L}^2(I_t)$ can be isometrically embedded into $\mathbb{H} := \mathbb{L}^2\left((0, L)\right)$ by setting $f(t) \equiv 0$ for $t \in (a_t, L)$ for any $f \in \mathbb{L}^2(I_t)$.

Define

$$\mathbb{X} := \left\{ f \in C\left([0, T]; \mathbb{H}\right), f(t) \in \mathbb{L}^2(I_t), \ \forall \ t \in [0, T] \right\}$$

equipped with the norm

$$\|f\|_{\mathbb{X}} = \sup_{0 \leq t \leq T} |f(t)|_t.$$

$\mathbb{X}$ is a Banach space, isometrically embedded into $C\left([0, T]; \mathbb{H}\right)$. Similarly, we can isometrically embed $\mathbb{H}_0^1(I_t)$ into $\mathbb{V} := \mathbb{H}_0^1\left((0, L)\right)$ by setting the functions in $\mathbb{H}_0^1(I_t)$ to be zero outside the interval I_t. This extension is well-defined since the functions in $\mathbb{H}_0^1(I_t)$ vanish at the boundary.

Assumptions and main results

We define the space:

$$\mathbb{Y} := \left\{ f \in \mathbb{L}^2\left([0,T]; \mathbb{H}_0^1((0,L))\right), f(t) \in \mathbb{H}_0^1(I_t) \text{ } a.e. \text{ } t \in [0,T] \right\}$$

equipped with the norm

$$\|f\|_{\mathbb{Y}}^2 = \int_0^T \|f(t)\|_t^2 \mathrm{d}t.$$

$\mathbb{Y}$ is a Hilbert space, isometrically embedded into $\mathbb{L}^2([0,T];\mathbb{V})$.
We will write $\|\cdot\|_{\mathbb{H}}$, $(\cdot,\cdot)_{\mathbb{H}}$, $\|\cdot\|_{\mathbb{V}}$, $(\cdot\ ,\cdot)_{\mathbb{V}}$ as $|\cdot|$, $(\cdot\ ,\cdot)$, $\|\cdot\|$ $\langle\cdot,\cdot\rangle$, respectively.
Analogously, we introduce the space:

$$\mathbb{Z} := \left\{ f \in \mathbb{L}^2([0,T];\mathbb{H}), f(t) \in \mathbb{L}^2(I_t) \text{ } a.e. \text{ } t \in [0,T] \right\}$$

equipped with the norm

$$\|f\|_{\mathbb{Z}}^2 = \int_0^T |f(t)|_t^2 \mathrm{d}t.$$

Assumptions and main results

For the Dirichlet Laplacian Δ_t on I_t, there exists an orthonormal eigensystem $\left\{\lambda_k(t), e_k(t)\right\}_{k\geq 1}$ on $\mathbb{H}_t$ given by

$$\lambda_k(t) = -\left(\frac{k\pi}{a_t}\right)^2; \quad e_k(t,x) = \sqrt{\frac{2}{a_t}} \sin\frac{k\pi x}{a_t}, \ x \in I_t. \tag{2}$$

$\{e_k(t)\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}_t$ and also an orthogonal basis in $\mathbb{V}_t$.

Take a sequence of independent standard real-valued Brownian motions $\{B^k\}_{k\geq 1}$ on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions, and an orthonormal basis $\{f_k\}_{k\geq 1}$ of $\mathbb{L}^2((0,L))$, then we introduce a cylindrical Brownian motion W on $\mathbb{H}$ given by

$$W_t := \sum_{k=1}^{\infty} f_k B_t^k, \ t \geq 0.$$

Now we give the assumptions on the mapping σ appearing in the stochastic heat equation (1). Let $\sigma : [0, T] \times \mathbb{H} \to \mathbb{L}_2(\mathbb{H}, \mathbb{H})$ be a Borel measurable mapping, where $\mathbb{L}_2(\mathbb{U}_1, \mathbb{U}_2)$ denotes the space of Hilbert-Schimdit operators from a Hilbert space $\mathbb{U}_1$ to another Hilbert space $\mathbb{U}_2$ equipped with the usual Hilbert-Schmidt norm $\|\cdot\|_{HS(\mathbb{U}_1,\mathbb{U}_2)}$. When there is no danger of causing ambiguity, we write $\|\cdot\|_{HS} = \|\cdot\|_{HS(\mathbb{U}_1,\mathbb{U}_2)}$.

Assumption II We assume that σ is measurable and there is a positive constant $K > 0$ such that, for any $s \in [0, T]$ and $u, v, h \in \mathbb{H}_s$,

(i) $\|\sigma(s, u) - \sigma(s, v)\|_{\mathrm{HS}} \leq K\|u - v\|_{\mathbb{H}}$;

(ii) $\|\sigma(s, u)\|_{\mathrm{HS}} \leq K(\|u\|_{\mathbb{H}} + 1)$;

(iii) $\sigma(s, h) \in \mathbb{L}_2(\mathbb{H}, \mathbb{H}_s)$.

The stochastic integral $\sigma\big(t, u(t)\big)\mathrm{d}W_t$ is understood as

$$\sigma\big(t, u(t)\big)\mathrm{d}W_t = \sum_{i=1}^{\infty} \sigma\big(t, u(t)\big) f_k \mathrm{d}B_t^k.$$

Denote the time-derivative $\partial_s \varphi(s, x)$ by $\varphi'(s, x)$. Recall the definition of $\mathcal{O}_T$ in Section 1, we have the following definition of the solution to (1).

Definition

$u \in \mathbb{X}$ *a.s.* is a solution of equation (1) if it is predictable and for $\forall\ t \in [0, T]$ and $\varphi \in \mathcal{C}_0^\infty(\bar{\mathcal{O}}_T) = \big\{g \in \mathcal{C}^\infty(\bar{\mathcal{O}}_T) :$ $g(t,0) = g(t,a_t) = 0, \forall t \in [0,T]\big\}$,

$$\begin{aligned}\int_0^{a_t} u(t,x)\varphi(t,x)\mathrm{d}x &- \int_0^{a_0} u_0(x)\varphi(0,x)\mathrm{d}x \\ &- \int_0^t \int_0^{a_s} u(s,x)\varphi'(s,x)\mathrm{d}x\mathrm{d}s \\ = \int_0^t \int_0^{a_s} u(s,x)\Delta_s\varphi(s,x)\mathrm{d}x\mathrm{d}s &+ \int_0^t \Big(\varphi(s), \sigma\big(s,u(s)\big)\mathrm{d}W_s\Big)_{\mathbb{H}} \ a.s.\end{aligned} \tag{3}$$

Now we are ready to state the main result.

Theorem

For a deterministic function $u_0 \in \mathbb{L}^2(I_0)$, there exists a unique solution $u \in \mathbb{L}^2(\Omega; \mathbb{X} \cap \mathbb{Y})$ to equation (1).

Approximating solutions

In this part, we will provide a number of uniform estimates for the approximating solutions. Moreover, we will show that the approximating solutions form a Cauchy sequence on a new probability space. This plays an important role in the proof of the main result.

To formulate the appropriate approximating equations, we start with some formal calculations. Expand the solution u (if it exists) of equation (1) with respect to the eigenbasis $e_k(s,x),\ k \in \mathbb{N}$ (see (2)) to get

$$u(s,x) = \sum_{k=1}^{\infty} \big(u(s), e_k(s)\big)_s e_k(s,x) = \sum_{k=1}^{\infty} A_k(s) e_k(s,x),$$

here

$$A_k(s) := \big(u(s), e_k(s)\big)_s = \int_0^{a_s} u(s,x) e_k(s,x) \mathrm{d}x.$$

Taking the eigenbasis $e_k(t,x)$, $k \in \mathbb{N}$ as the test functions in (3), it follows that

$$A_k(t) - (u_0, e_k(0))_0 - \int_0^t \int_0^{a_s} u(s,x) e_k'(s,x) \mathrm{d}x \mathrm{d}s$$
$$= \int_0^t \lambda_k(s) A_k(s) \mathrm{d}s + \int_0^t \Big(e_k(s), \sigma(s, u(s)) \mathrm{d}W_s \Big)_{\mathbb{H}}.$$

A formal calculation yields that

$$A_k(t) - (u_0, e_k(0))_0 - \sum_{j=1}^{\infty} \int_0^t A_j(s) b_{jk}(s) \mathrm{d}s \tag{4}$$
$$= \int_0^t \lambda_k(s) A_k(s) \mathrm{d}s + \int_0^t \Big(e_k(s), \sigma(s, \sum_{j=1}^{\infty} A_j(s) e_j(s)) \mathrm{d}W_s \Big)_{\mathbb{H}},$$

where

$$b_{jk}(s) = \int_0^{a_s} e_j(s,x) e_k'(s,x) \mathrm{d}x = \begin{cases} (-1)^{j+k} \cdot \frac{a_s'}{a_s} \cdot \frac{2jk}{j^2 - k^2}, & j \neq k, \\ 0, & j = k. \end{cases}$$

Approximating solutions

This suggests to consider the following interacting systems of stochastic differential equations: for $k = 1, \cdots, n$,

$$A_k^n(t) - (u_0, e_k(0))_0 - \sum_{j=1}^{n} \int_0^t A_j^n(s) b_{jk}(s) \mathrm{d}s \tag{6}$$

$$= \int_0^t \lambda_k(s) A_k^n(s) \mathrm{d}s + \int_0^t \Big(e_k(s), \sigma\big(s, \sum_{j=1}^{n} A_j^n(s) e_j(s)\big) \mathrm{d}W_s \Big)_{\mathbb{H}}.$$

Note that the SDE (6) has a unique $\mathbb{R}^n$-valued continuous solution $\big(A_1^n(t), A_2^n(t), ..., A_n^n(t)\big)$ since all the coefficients are Lipschitz continuous. Set

$$u^n(t,x) := \sum_{k=1}^{n} A_k^n(t) e_k(t,x). \tag{7}$$

Approximating solutions

We begin with some moment estimates of $\{u^n\}_{n\geq 1}$ in the space $\mathbb{X} \cap \mathbb{Y}$.

Proposition 1 There exists a positive constant $C_1 > 0$ such that

$$\sup_{n\in\mathbb{N}} E\Big[\sup_{0\leq t\leq T} \big|u^n(t)\big|_t^2 + \int_0^T \big\|u^n(t)\big\|_t^2 \mathrm{d}t\Big] \leq C_1.$$

The Proposition is proved using essentially the skew-symmetry of the matrix $b_{jk}(s)$ and the Ito formula for $A_k^n(t)$.

Approximating solutions

We also have the uniform estimate of higher order moments.

Corollary 2 For all $p > 1$ there exists a positive constant $C_{p,K,T}$ such that

$$\sup_{n\in\mathbb{N}} E\Big\{ \sup_{0\le t\le T} \big|u^n(t)\big|_t^{2p} \Big\} \le C_{p,K,T},$$

$$\sup_{n\in\mathbb{N}} E\Big\{ \Big(\int_0^T \|u^n(t)\|_t^2 \mathrm{d}t \Big)^p \Big\} \le C_{p,K,T}.$$

To prove the tightness of the laws of $u^n, n \geq 1$, we recall the following lemma characterizing the compact subsets of $\mathbb{Z}$ proved in [3].

Let J_i denote a family of equicontinuous real-valued functions on $[0, T]$. For a positive constant N, set $K_{N,J} := \cap_{i=1}^{\infty} K_{N,J_i}$, where

$$K_{N,J_i} = \Big\{ g \in \mathbb{X} \cap \mathbb{Y} : \sup_{0 \leq t \leq T} |g(t)|_t \leq N, \int_0^T \|g(t)\|_t^2 \mathrm{d}t \leq N,$$
$$g_i = \left\{ g_i(t) := \big(g(t), e_i(t)\big)_t, \ t \in [0, T] \right\} \in J_i \Big\}.$$

Lemma 3. $K_{N,J}$ is precompact in $\mathbb{Z}$.

Approximating solutions

The next result gives the tightness of $\{\mathcal{L}(u^n)\}_{n\geq 1}$, the family of distributions of $\{u^n\}_{n\in\mathbb{N}}$.

Proposition 4. $\{\mathcal{L}(u^m)\}_{m\geq 1}$ is tight in $\mathbb{Z}$.

Approximating solutions

By a generalized Skorokhod representation theorem, there exists a new probability space $(\Omega^*, \mathcal{F}^*, P^*)$, a sequence of $\mathbb{Z}$-valued random processes $\{u_*, u_*^m, m \geq 1\}$ and an $\mathbb{H}$-cylindrical Brownian motion W^* such that

$$P^* \circ (u_*^m, W^*)^{-1} = P \circ (u^m, W)^{-1} \tag{8}$$

and that, taking a subsequence if necessary, $\lim_{m\to\infty} u_*^m = u_*$ in $\mathbb{Z}$, P^*-*a.s*. Moreover, we have the following stronger convergence result which will be used later.

Lemma 5. $\{u_*^n\}_{n\geq 1}$ is a Cauchy sequence in probability in the space $\mathbb{X} \cap \mathbb{Y}$.

Sketch of the proof

We can assume $m > n$. Set

$$A_k^{n,*}(t) := \big(u_*^n(t), e_k(t)\big)_t. \tag{9}$$

Let $k \le n$, we have

$$\begin{aligned}
& A_k^{m,*}(t) - A_k^{n,*}(t) \\
= \ & \sum_{j=1}^{n} \int_0^t \Big(A_j^{m,*}(s) - A_j^{n,*}(s)\Big) b_{jk}(s)\mathrm{d}s + \sum_{j=n+1}^{m} \int_0^t A_j^{m,*}(s) b_{jk}(s)\mathrm{d}s \\
& + \int_0^t \lambda_k(s)\big(A_k^{m,*}(s) - A_k^{n,*}(s)\big)\mathrm{d}s \\
& + \int_0^t \Big(e_k(s), \big(\sigma\big(s, u_*^m(s)\big) - \sigma\big(s, u_*^n(s)\big)\big)\mathrm{d}W_s^*\Big).
\end{aligned}$$

The following relationship holds.

$$\left|u_*^m(t) - u_*^n(t)\right|_t^2 = \sum_{k=1}^{n} \left|A_k^{m,*}(t) - A_k^{n,*}(t)\right|^2 + \sum_{k=n+1}^{m} \left|A_k^{m,*}(t)\right|^2, \tag{10}$$

$$\begin{aligned} \left\|u_*^m(t) - u_*^n(t)\right\|_t^2 &= -\sum_{k=1}^{n} \lambda_k(t)\left|A_k^{m,*}(t) - A_k^{n,*}(t)\right|^2 \\ &\quad - \sum_{k=n+1}^{m} \lambda_k(t)\left|A_k^{m,*}(t)\right|^2. \end{aligned}$$

Sketch of the proof

The proof of this lemma is divided into two steps.
Step 1. We will prove that

$$\sup_{0\le t\le T}\sum_{k=1}^{n}|A_k^{m,*}(t)-A_k^{n,*}(t)|^2-2\sum_{k=1}^{n}\int_0^T\lambda_k(s)\big(A_k^{m,*}(s)-A_k^{n,*}(s)\big)^2\mathrm{d}s \rightarrow 0 \tag{11}$$

in probability as $m,n\rightarrow\infty$.

For $l\ge 1$, denote a time-dependent orthogonal projection by

$$P_l^s\big(u(s)\big)=\sum_{i=1}^{l}\big(u(s),e_i(s)\big)_s e_i(s).$$

By Itô's formula,

$$
\begin{aligned}
&\sum_{k=1}^{n} |A_k^{m,*}(t) - A_k^{n,*}(t)|^2 - 2\sum_{k=1}^{n}\int_0^t \lambda_k(s)\big(A_k^{m,*}(s) - A_k^{n,*}(s)\big)^2 \mathrm{d}s \\
&= 2\sum_{k=1}^{n}\sum_{j=1}^{n}\int_0^t \big(A_j^{m,*}(s) - A_j^{n,*}(s)\big) b_{jk}(s)\big(A_k^{m,*}(s) - A_k^{n,*}(s)\big)\mathrm{d}s \\
&\quad + 2\sum_{k=1}^{n}\sum_{j=n+1}^{m}\int_0^t A_j^{m,*}(s) b_{jk}(s)\big(A_k^{m,*}(s) - A_k^{n,*}(s)\big)\mathrm{d}s \\
&\quad + 2\int_0^t \left(P_n^s\big(u_*^m(s) - u_*^n(s)\big), \Big(\sigma\big(s, u_*^m(s)\big) - \sigma\big(s, u_*^n(s)\big)\Big)\mathrm{d}W_s^*\right) \\
&\quad + \int_0^t \sum_{k=1}^{n}\sum_{j=1}^{\infty} \left|\sigma_j^k\big(s, u_*^m(s)\big) - \sigma_j^k\big(s, u_*^n(s)\big)\right|^2 \mathrm{d}s \\
&=: \mathrm{I}_n^m(t) + \mathrm{II}_n^m(t) + \mathrm{III}_n^m(t) + \mathrm{IV}_n^m(t). \qquad (12)
\end{aligned}
$$

Sketch of the proof

Since $b_{jk}(s) = -b_{kj}(s)$,

$$\mathrm{I}_n^m(t) = 0. \tag{13}$$

For the second term, we have

$$\begin{aligned}
&\left|\mathrm{II}_n^m(t)\right| \\
&= 2\Big|\sum_{k=1}^{n}\sum_{j=n+1}^{m}\int_0^t A_j^{m,*}(s)\frac{a_s'}{a_s}\cdot\frac{2jk}{j^2-k^2}\big(A_k^{m,*}(s)-A_k^{n,*}(s)\big)(-1)^{j+k}\mathrm{d}s\Big| \\
&= 2\Big|\int_0^t\sum_{k=1}^{n}\sum_{j=n+1}^{m}\frac{A_j^{m,*}(s)j\pi}{a_s}\cdot\frac{\big(A_k^{m,*}(s)-A_k^{n,*}(s)\big)k\pi}{a_s} \\
&\qquad\qquad\cdot\frac{2a_s a_s'}{\pi^2(j^2-k^2)}(-1)^{j+k}\mathrm{d}s\Big| \\
&\le \cdots
\end{aligned}$$

Sketch of the proof

$$
\begin{aligned}
&\leq 2\int_0^t \|u_*^m(s)\|_s \Big(-\sum_{k=1}^{n} \big|A_k^{m,*}(s) - A_k^{n,*}(s)\big|^2 \lambda_k(s)\Big)^{\frac{1}{2}} \\
&\qquad \times \Big(\sum_{j=n+1}^{m} \sum_{k=1}^{n} \frac{1}{(j^2-k^2)^2}\Big)^{\frac{1}{2}} \cdot \frac{2|a_s'|a_s}{\pi^2} \mathrm{d}s \\
&\leq 2\int_0^t \|u_*^m(s)\|_s \|u_*^m(s) - u_*^n(s)\|_s \Big(\sum_{j=n+1}^{m} \sum_{k=1}^{n} \frac{1}{(j^2-k^2)^2}\Big)^{\frac{1}{2}} \cdot \frac{2|a_s'|a_s}{\pi^2} \mathrm{d}s \\
&\leq \frac{4L^2}{\pi^2} \left(\sum_{j=n+1}^{m} \sum_{k=1}^{n} \frac{1}{(j^2-k^2)^2}\right)^{\frac{1}{2}} \|u_*^m\|_{\mathbb{Y}} \|u_*^m - u_*^n\|_{\mathbb{Y}}.
\end{aligned}
$$

Sketch of the proof

We have

$$E^*\Big[\sup_{0\le t\le T}\Big|\mathrm{II}_n^m(t)\Big|\Big] \le 2C_1 C_{K,L}\Big(\sum_{j=n+1}^{m}\sum_{k=1}^{n}\frac{1}{(j^2-k^2)^2}\Big)^{\frac{1}{2}}.$$

Now, setting $l = n+1-k$, we see that

$$\begin{aligned}\sum_{j=n+1}^{m}\sum_{k=1}^{n}\frac{1}{(j^2-k^2)^2} &= \sum_{j=n+1}^{m}\sum_{k=1}^{n}\frac{1}{(j+k)^2(j-k)^2} \\ &\le \sum_{j=n+1}^{m}\frac{1}{j^2}\sum_{k=1}^{n}\frac{1}{(j-k)^2} \\ &\le \sum_{j=n+1}^{m}\frac{1}{j^2}\sum_{l=1}^{n}\frac{1}{l^2} \to 0 \ as\ m,\ n\ \to\infty.\end{aligned}$$

Sketch of the proof

It follows that

$$E^*\left[\sup_{0\leq t\leq T}\left|\mathrm{II}_n^m(t)\right|\right] \to 0 \text{ } as \text{ } m,\ n \ \to \infty.$$

In particular,

$$\sup_{0\leq t\leq T}\left|\mathrm{II}_n^m(t)\right| \to 0 \text{ in probability as } m, n \to \infty. \qquad (14)$$

Sketch of the proof

We can also show that

$$\sup_{0\le t\le T}\left|III_n^m(t)\right|,\ \sup_{0\le t\le T}\left|IV_n^m(t)\right|\to 0 \tag{15}$$

in probability as $m,n\to\ \infty$. Putting the above convergence together, we obtain that

$$\sup_{0\le t\le T}\sum_{k=1}^{n}|A_k^{m,*}(t)-A_k^{n,*}(t)|^2-2\sum_{k=1}^{n}\int_0^T\lambda_k(s)\left|A_k^{m,*}(s)-A_k^{n,*}(s)\right|^2\mathrm{d}s$$
$$\to\ 0$$

as $m,n\to\infty$ in probability.

Step 2. We prove that the remaining terms also tend to zero, i.e.,

$$\sup_{0\le t\le T}\sum_{k=n}^{m}|A_k^{m,*}(t)|^2 - 2\sum_{k=n}^{m}\int_0^T \lambda_k(s)\left|A_k^{m,*}(s)\right|^2 \mathrm{d}s \to \ 0 \quad (16)$$

as $m, n \to \infty$ in probability.

Combining step 1 and step 2, we see that $\{u_*^n\}_{n\ge 1}$ is a Cauchy sequence in probability in the space $\mathbb{X}\cap\mathbb{Y}$. Hence, we can assume that $u_* \in \mathbb{X}\cap\mathbb{Y}$ and there exists a subsequence n_k such that $\lim_{k\to\infty} u_*^{n_k} = u_*$ in $\mathbb{X}\cap\mathbb{Y}$ a.s.

Sketch of the proof of the main result

The limit $u_*(t,x)$ obtained above admits the following expansion

$$u_*(t,x) = \sum_{k=1}^{\infty} A_k^*(t) e_k(t,x), \tag{17}$$

where the series converges in $\mathbb{X}$. Now we will show that u_* is indeed a solution of the stochastic heat equation (1). Passing to the limit, we can show that the stochastic processes $A_k^*, k \geq 1$ satisfy the following infinite dimensional system: for any $k \geq 1$ and $t \geq 0$, a.s.

$$A_k^*(t) - (u_0, e_k(0))_0 - \sum_{j=1}^{\infty} \int_0^t A_j^*(s) b_{jk}(s) \mathrm{d}s \tag{18}$$

$$= \int_0^t \lambda_k(s) A_k^*(s) \mathrm{d}s + \int_0^t \Big(e_k(s), \sigma\big(s, u_*(s)\big) \mathrm{d}W_s^* \Big).$$

Proposition 6 The random field u_* obtained above is a solution to the stochastic heat equation (1), namely, for all $t \in [0, T]$ and $\varphi \in \mathcal{C}_0^\infty(\bar{\mathcal{O}}_T)$,

$$\int_0^{a_t} u_*(t,x)\varphi(t,x)\mathrm{d}x - \int_0^{a_0} u_0(x)\varphi(0,x)\mathrm{d}x - \int_0^t \int_0^{a_s} u_*(s,x)\varphi'(s,x)\mathrm{d}x\mathrm{d}s$$
$$= \int_0^t \int_0^{a_s} u_*(s,x)\Delta\varphi(s,x)\mathrm{d}x\mathrm{d}s + \int_0^t \Big(\varphi(s), \sigma\big(s, u_*(s)\big) dW_s^*\Big) \; a.s. \tag{19}$$

Let φ be a test function, i.e. $\varphi \in \mathcal{C}_0^\infty(\bar{\mathcal{O}}_T)$ and define $\varphi_k(s) := \big(\varphi(s), e_k(s)\big)_s$. Since $e_k(s,x)$ and $\varphi(s,x)$ vanish at the boundary of I_s, we have

$$\mathrm{d}\varphi_k(s) = \big(\varphi'(s), e_k(s)\big)_s \mathrm{d}s + \big(\varphi(s), e_k'(s)\big)_s \mathrm{d}s,$$

and

$$\begin{aligned}
&\varphi_k A_k^*(t) \\
&= \varphi_k (u_0, e_k(0))_0 + \int_0^t \varphi_k(s)\big(u_*(s), e_k'(s)\big)_s \mathrm{d}s + \int_0^t \lambda_k(s)\varphi_k(s)A_k^*(s)\mathrm{d}s \\
&\quad + \int_0^t \Big(\varphi_k(s)e_k(s), \sigma\big(s, u_*(s)\big)\mathrm{d}W_s^*\Big) + \int_0^t A_k^*(s)\big(\varphi'(s), e_k(s)\big)_s \mathrm{d}s \\
&\quad + \int_0^t A_k^*(s)\big(\varphi(s), e_k'(s)\big)_s \mathrm{d}s \\
&= \mathbb{I}_k + \mathbb{II}_k(t) + \mathbb{III}_k(t) + \mathbb{IV}_k(t) + \mathbb{V}_k(t) + \mathbb{VI}_k(t). \qquad (20)
\end{aligned}$$

Recall $\left(u_*(t), \varphi(t)\right)_t = \sum_{k=1}^{\infty} \varphi_k A_k^*(t)$. Adding up (20) to an arbitrarily big natural number n and then letting $n \to \infty$, we obtain

(i)

$$\sum_{k=1}^{\infty} \mathbb{I}_k = \left(\varphi(0), u(0)\right)_0,$$

(ii)

$$\sum_{k=1}^{\infty} \mathbb{III}_k(t) = \int_0^t \Delta\varphi(s) u_*(s) \mathrm{d}s \ a.s.,$$

Sketch of the proof of the main result

(iii)

$$\sum_{k=1}^{\infty} \mathbb{IV}_k(t) = \int_0^t \Big(\varphi(s), \sigma\big(s, u_*(s)\big)\mathrm{d}W_s^*\Big) \text{ in probability,}$$

(iv)

$$\sum_{k=1}^{\infty} \mathbb{V}_k(t) = \int_0^t \big(\varphi'(s), u_*(s)\big)_s \mathrm{d}s \text{ in } \mathbb{L}^2(\Omega^*).$$

Sketch of the proof of the main result

On the other hand, we note that

$$\mathbb{III}_k(t) = \int_0^t \varphi_k(s)\big(u_*(s), e_k'(s)\big)_s \mathrm{d}s = \int_0^t \varphi_k(s) \sum_{j=1}^{\infty} A_j^*(s) b_{jk}(s) \mathrm{d}s.$$

$$\mathbb{VI}_k(t) = \int_0^t A_k^*(s)\big(\varphi(s), e_k'(s)\big)_s \mathrm{d}s = \int_0^t A_k^*(s) \sum_{j=1}^{\infty} \varphi_j(s) b_{jk}(s) \mathrm{d}s.$$

Since $\{b_{jk}(s)\}_{j,k}$ is skew-symmetric with respect to (j,k), we see that

$$\sum_{k=1}^{\infty} \mathbb{III}_k(t) + \sum_{k=1}^{\infty} \mathbb{VI}_k(t) = 0.$$

Sketch of the proof of the main result

The interchange of the infinite sum with the integral can be justified because we can show that

$$
\begin{aligned}
& E^*\Big[\int_0^T \sum_{k,j} |\varphi_k(s)||A_j^*(s)||b_{jk}(s)|\mathrm{d}s\Big] \qquad (21)\\
= & E^*\Big[\int_0^t \sum_{k\neq j} |\varphi_k(s)|\frac{k}{|a_s|}|A_j^*(s)|\frac{j}{|a_s|}\frac{|a_s||a_s'|}{|j^2-k^2|}\mathrm{d}s\Big]\\
\leq & CE^*\Big[\int_0^t \|\varphi(s)\|_s^2\mathrm{d}s\Big]^{\frac{1}{2}}E^*\Big[\int_0^t \|u_*(s)\|_s^2\mathrm{d}s\Big]^{\frac{1}{2}}\\
< & \infty.
\end{aligned}
$$

Putting the above equations together we finally arrive at

$$\int_0^{a_t} u_*(t,x)\varphi(t,x)\mathrm{d}x - \int_0^{a_0} u_0(x)\varphi(0,x)\mathrm{d}x - \int_0^t \int_0^{a_s} u_*(s,x)\varphi'(s,x)\mathrm{d}x\mathrm{d}s = \int_0^t \int_0^{a_s} u_*(s,x)\Delta_s\varphi(s,x)\mathrm{d}x\mathrm{d}s + \int_0^t \Big(\varphi(s), \sigma\big(s, u_*(s)\big)\mathrm{d}W_s^*\Big)_{\mathbb{H}} \ a.s., \tag{22}$$

completing the proof.

Next result is an energy identity/Itô-type formula for the solution which will be used to prove the pathwise uniqueness.

Proposition 7. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}_{t\geq 0}, \tilde{P}, \tilde{W}, \tilde{u})$ be a solution to equation (1). We have

$$
\begin{aligned}
|\tilde{u}(t)|_t^2 &= |u(0)|_0^2 - 2\int_0^t \|\tilde{u}(s)\|_s^2 \mathrm{d}s + 2\int_0^t \Big(\tilde{u}(s), \sigma(s, \tilde{u}(s))\mathrm{d}\tilde{W}_s\Big) \\
&\quad + \int_0^t \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} \left|\sigma_j^k(s, \tilde{u}(s))\right|^2 \mathrm{d}s \\
&= |u(0)|_0^2 - 2\int_0^t \|\tilde{u}(s)\|_s^2 \mathrm{d}s + 2\int_0^t \Big(\tilde{u}(s), \sigma(s, \tilde{u}(s))\mathrm{d}\tilde{W}_s\Big) \\
&\quad + \int_0^t \left\|\sigma(s, \tilde{u}(s))\right\|_{\mathrm{HS}}^2 \mathrm{d}s.
\end{aligned}
$$

Sketch of the proof of the main result

Here is the uniqueness of the solution.
Theorem 8. The solution of the stochastic heat equation (1) is pathwise unique in the space $\mathbb{X} \cap \mathbb{Y}$.

Completion of the proof of the main result
Proposition 7 gives the existence of a probabilistic weak solution. Now The main result follows from the pathwise uniqueness proved in Theorem 8 and the well-known Yamada-Watanabe theorem.

Sketch of the proof of the main result

Given the existence of a unique solution u of equation (1), the next result shows that the solution u can be approximated by solutions $u^n(t)$ of the finite dimensional systems (6) or (7).

Proposition 9. The solutions $\{u^n\}_{n\geq 1}$ of the finite-dimensional interacting systems in (6) converge to u in $L^2(\Omega; \mathbb{X} \cap \mathbb{Y})$ as $n \to \infty$.

Future work

- The regularity of the solution as a function of the space and time, e.g., the Holder continuity, differentiability etc
- The long time behavior of the solution, i.g., existence of stationary solution, ergodicity etc

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